Mean-field theory for a passive scalar advected by a turbulent velocity field with a random renewal time

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Mean-field theory for turbulent transport of a passive scalar (e.g., particles and gases) is discussed. Equations for the mean number density of particles advected by a random velocity field, with a finite correlation time, are derived. Mean-field equations for a passive scalar comprise spatial derivatives of high orders due to the nonlocal nature of passive scalar transport in a random velocity field with a finite correlation time. A turbulent velocity field with a random renewal time is considered. This model is more realistic than that with a constant renewal time used by Elperin *et al*. [Phys. Rev. E **61**, 2617 (2000)], and employs two characteristic times: the correlation time of a random velocity field τ_c, and a mean renewal time τ. It is demonstrated that the turbulent diffusion coefficient is determined by the minimum of the times τ_c and τ. The mean-field equation for a passive scalar was derived for different ratios of τ/τ_c. The important role of the statistics of the field of Lagrangian trajectories in turbulent transport of a passive scalar, in a random velocity field with a finite correlation time, is demonstrated. It is shown that in the case $\tau_c \ll \tau \ll \tau_N$ the form of the mean-field equation for a passive scalar is independent of the statistics of the velocity field, where τ_N is the characteristic time of variations of a mean passive scalar field.

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I. INTRODUCTION

Passive scalar transport in a turbulent velocity field was studied intensively during the last years (see, e.g., Refs. [1–7]). The main progress was achieved in understanding the properties of the high-order moments of a passive scalar, e.g., non-Gaussian statistics and anomalous scalings. Note that the non-Gaussian properties of passive fields advected by turbulent fluid flow were already discussed in Ref. [8].

While the analysis of high-order moments of a passive scalar was the subject of numerous studies, only a few studies discussed the dynamics of a mean passive scalar field (see, e.g., Refs. [9,10]). However, the mean-field theory for turbulent transport of particles and gases is of great importance in view of numerous applications. In particular, this theory is applied for an analysis of transport of aerosols, pollutants and cloud droplets in atmospheric turbulence of the Earth and other planets (see, e.g., Refs. [11–20]), dust transfer in interstellar turbulence and turbulent transport of particles and gases in industrial flows (see, e.g., Refs. [21–24]).

The mean-field equation for a passive scalar is given by

$$\frac{\partial N}{\partial t} + \boldsymbol{\nabla} \cdot (N\mathbf{V}_{\rm eff} - D_T \boldsymbol{\nabla} N) = 0, \tag{1}$$

where D_T is the turbulent diffusion coefficient, and $\mathbf{V}_{\rm eff}$ is an effective velocity. For an incompressible velocity field $\mathbf{V}_{\rm eff} = \mathbf{U}$, where $\mathbf{U}$ is the mean fluid velocity (see, e.g., Refs. [25–27]). However, when the velocity field $\mathbf{v}$ is not divergence free (e.g., due to particle's inertia) the effective velocity is given by

$$\mathbf{V}_{\rm eff} = \mathbf{U} - \langle \tau_c \mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) \rangle, \tag{2}$$

(see Refs. [6,28]) where τ_c is the correlation time of a turbulent velocity field. The second term in Eq. (2) can be interpreted as an additional turbulent flux of particles, i.e., the total turbulent flux of particles is given by

$$\mathbf{J}_T = N\mathbf{V}_{\rm eff} - D_T \boldsymbol{\nabla} N, \tag{3}$$

and $\langle \tau_c \mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) \rangle \propto \boldsymbol{\nabla} T$, where T is the mean fluid temperature. The additional turbulent nondiffusive flux of particles [which is caused by the term $\propto \langle \tau_c \mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) \rangle$] results in the formation of inhomogeneities of particle distributions due to the excitation of a large-scale instability. One of the most important conditions for instability is the inhomogeneous spatial distribution of the mean temperature. In particular, large-scale instability can be excited in the vicinity of the minimum in the mean temperature (see Refs. [6,28]). For instance, in atmospheric turbulence large-scale instability can be excited in the vicinity of the temperature inversion (see Ref. [18]).

Equations (1) and (2) were derived for the random velocity field δ correlated in time. It was recently shown Ref. [10] that when a random velocity field has a small yet finite correlation time, the mean-field equation has a much more complicated form than Eq. (1). In particular, it contains high-order spatial derivatives. In Ref. [10] a model of a random compressible velocity field with a constant renewal time was considered. This allowed one derive a mean-field equation for the number density of particles advected by a random velocity field with a finite correlation time. However, this model is limited by the assumption of a constant renewal time, and does not describe a real turbulent velocity field.

In the present study we consider a turbulent velocity field with a random renewal time. This model is more realistic than that used in Ref. [10]. In particular, when a mean renewal time is very large in comparison with the correlation time of the velocity field, we recover the results obtained for a turbulent flow without renewal. In this model there are two characteristic times: the correlation time of a random velocity field τ_c, and the mean renewal time τ. It is demonstrated that the turbulent diffusion coefficient is determined by the minimum of the times τ_c and τ. In the present study the mean-field equation for a passive scalar was derived for different ratios of τ/τ_c. We demonstrate an important role of the statistics of the field of Lagrangian trajectories in turbulent transport of a passive scalar in a random velocity field with a finite correlation time.

The model of a turbulent velocity field with a random renewal time reproduces important features of some real turbulent flows. Indeed, the interstellar turbulence is driven by supernova explosions and it loses memory at the instants of explosions (see, e.g., Ref. [29]), which can be described by the Poisson process. Note that discussions on practical diffusion models in atmospheric sciences based on Lagrangian techniques can be found, e.g., in reviews Refs. [16,20].

II. GOVERNING EQUATIONS

In the present study we derive an equation for a mean passive scalar field (e.g., a number density of particles) advected by a random velocity field. The equation for the number density $n(t,\mathbf{r})$ of small particles advected by a random fluid flow reads

$$\frac{\partial n(t,\mathbf{r})}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{v}) = D\Delta n, \tag{4}$$

where D is the coefficient of molecular (Brownian) diffusion, and $\mathbf{v}(t,\mathbf{r})$ is a random velocity field of particles which they acquire in a turbulent fluid velocity field. We will take into account the compressibility of velocity field $\boldsymbol{\nabla}\cdot\mathbf{v}\neq 0$, which occurs due either to the compressibility of the fluid itself or due to the particle's inertia. Equation (4) implies the conservation of the total number of particles in a closed volume.

When $D\neq 0$, particles are transported by both fluid advection and Brownian motions (i.e., molecular diffusion). In this case the function $n(t,\mathbf{x})$ is given by

$$n(t,\mathbf{x}) = M_{\boldsymbol{\xi}}\{G(t,\boldsymbol{\xi})\exp(\boldsymbol{\xi}^{*}\cdot\boldsymbol{\nabla})n(s,\mathbf{x})\} \tag{5}$$

(see Appendix A), where $M_{\boldsymbol{\xi}}\{\cdot\}$ denotes the mathematical expectation over the Wiener paths

$$\boldsymbol{\xi} = \mathbf{x} - \int_0^{t-s}\mathbf{v}(t-\mu,\boldsymbol{\xi})\,d\mu + \sqrt{2D}\mathbf{w}(t-s). \tag{6}$$

$\boldsymbol{\xi}^{*} = \boldsymbol{\xi} - \mathbf{x}$, $\mathbf{w}(t)$ is a Wiener process which describes the molecular diffusion, $G(t,\mathbf{x}) = \exp[-\int_s^t b(\mu,\boldsymbol{\xi})\,d\mu]$ and $b = \boldsymbol{\nabla}\cdot\mathbf{v}$. Note that there exists a large number of studies (particularly Lagrangian models of passive tracer transport in the atmospheric and oceanic sciences), where Eq. (4) is solved numerically using averaging over trajectory (6) (see, e.g., Refs. [16,20]).

We consider a random velocity field which loses memory at random instants of time. The velocity fields before and after renewal are assumed to be statistically independent. We introduce this assumption in order to decouple averaging into averagings over two time intervals. Indeed, the function $G(t,\boldsymbol{\xi})$ in Eq. (5) is determined by the velocity field after the renewal, while the number density of particles $n(s,\mathbf{x})$ is determined by the velocity field before renewal. This model reproduces important features of some real turbulent flows. Thus, e.g., the interstellar turbulence, which is driven by supernova explosions, loses memory in the instants of explosions (see, e.g., Ref. [29]). Between renewals the velocity field can be random, with intrinsic statistics. In order to obtain a statistically stationary random velocity field, we assume that velocity fields between renewals have the same statistics. The random renewal instants destroy the stationarity of the velocity field. On the other hand, between random renewal instants the velocity field is stationary.

In order to perform calculations in a closed form, we assume that random renewal times can be described by a Poisson process, as suggested in Ref. [29]. A two-point correlation function of the velocity field is given by

$$\hat{f}_{mn} = \langle v_m(t,\mathbf{x})v_n(s,\mathbf{y})\rangle = f_{mn}(t-s,\mathbf{x},\mathbf{y})\exp(-\lambda|t-s|), \tag{7}$$

where the function $\exp(-\lambda|t-s|)$ describes a Poisson process for a random renewal time, and $\tau = 1/\lambda$ is the mean renewal time. The correlation time of a random velocity field is given by

$$\tau_c = \int f_{mm}(\tau,\mathbf{x},\mathbf{x})\,d\tau / f_{mm}(0,\mathbf{x},\mathbf{x}). \tag{8}$$

Thus we introduced a model which is described by three random processes: (i) the Wiener random process, which describes Brownian motions, i.e., molecular diffusion; (ii) the Poisson process for random renewal times; and (iii) the random velocity field between the renewals. The correlation time in the maximum scale of turbulent motions l_0 is $\tau_0 = l_0/u_0$, where u_0 is the characteristic velocity in the maximum scale of turbulent motions l_0. The model of a turbulent velocity field with random renewal instants used in our paper implies that a random forcing acts continuously. Therefore, between renewals, turbulence does not decay [see Eq. (7) of our paper].

Now we average Eq. (5) over a random velocity field for a given realization of a Poisson process,

$$\langle n(t,\mathbf{x})\rangle = M_{\boldsymbol{\xi}}\{\langle G(t',\boldsymbol{\xi})\exp[\boldsymbol{\xi}^{*}(t')\cdot\boldsymbol{\nabla}]\rangle\langle n(s,\mathbf{x})\rangle\}, \tag{9}$$

where the time s is the last renewal time before t, and $t' = t-s$ is a random variable. Indeed, averaging of the functions $G(t',\boldsymbol{\xi})\exp[\boldsymbol{\xi}^{*}(t')\cdot\boldsymbol{\nabla}]$, $n(s,\mathbf{x})$ can be decoupled into the product of averages since the first function is determined by the velocity field after the renewal, while the second function $n(s,\mathbf{x})$ is determined by the velocity field before renewal. We average Eq. (9) over the random renewal times t', taking into account the stationarity of the Poisson process. The probability density $p(t')$ for a random renewal time t' is given by $p(t')=\lambda\exp(-\lambda t')$. The resulting averaged equation is given by

$$N(t,\mathbf{x})=\int_0^t P(\mu,\mathbf{x},i\boldsymbol{\nabla})N(t-\mu,\mathbf{x})\lambda\exp(-\lambda\mu)d\mu + \exp(-\lambda t)P(t,\mathbf{x},i\boldsymbol{\nabla})N_0(\mathbf{x}), \qquad (10)$$

where $N(t,\mathbf{x})=E_\tau\{\langle n(t,\mathbf{x})\rangle\}$ and $N_0(\mathbf{x})=N(t=0,\mathbf{x})$, $E_\tau\{\cdot\}$ denotes an averaging over random renewal times,

$$P(t,\mathbf{x},i\boldsymbol{\nabla})=M_{\boldsymbol{\xi}}\{\langle G(t,\boldsymbol{\xi})\exp[\boldsymbol{\xi}^*\cdot\boldsymbol{\nabla}\rangle]\}, \qquad (11)$$

and the probability that no renewal occurs during time t is $\exp(-\lambda t)$. The first term in Eq. (10) describes the case when there is at least one renewal of the velocity field during the time t (i.e., the Poisson event), whereas the second term describes the case when there is no renewal during the time t. Now we use the identity

$$N(t-\mu,\mathbf{x})=\exp\left(-\mu\frac{\partial}{\partial t}\right)N(t,\mathbf{r}),$$

which follows from the Taylor expansion

$$f(t+\tau)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\tau\frac{\partial}{\partial t}\right)^m f(t)=\exp\left(\tau\frac{\partial}{\partial t}\right)f(t).$$

Thus Eq. (10) can be rewritten as

$$\left[\int_0^{\lambda t}P\left(\frac{\sigma}{\lambda},\mathbf{x},i\boldsymbol{\nabla}\right)\exp\left(-\frac{\sigma}{\lambda}\frac{\partial}{\partial t}\right)\exp(-\sigma)d\sigma-1\right]N(t,\mathbf{x}) + \exp(-\lambda t)P(t,\mathbf{x},i\boldsymbol{\nabla})N_0(\mathbf{x})=0. \qquad (12)$$

Equation (12) generalizes the mean-field equation for a passive scalar advected by a random velocity field with a constant renewal time [compare with Eqs. (19) and (D6) derived in Ref. [10])]. Indeed, when $\tau=$const. Eq. (12) reads

$$\left[P(\tau,\mathbf{x},i\boldsymbol{\nabla})\exp\left(-\tau\frac{\partial}{\partial t}\right)-1\right]N(t,\mathbf{x})=0. \qquad (13)$$

III. MEAN-FIELD EQUATION FOR A PASSIVE SCALAR

In this section we derive mean-field equations for a passive scalar using different models of a random velocity field. This allows us to elucidate some important features of a turbulent transport of a passive scalar in a random velocity field with a finite correlation time.

A. Random velocity field with a small renewal time

Consider a very small renewal time, i.e., $\tau\ll\min\{\tau_c;\tau_0\}$. Expanding the functions $P(\sigma/\lambda,\mathbf{x},i\boldsymbol{\nabla})$ and $\exp[-(\sigma/\lambda)(\partial/\partial t)]$ into Taylor series in the vicinity $\sigma/\lambda=0$ yields

$$\int_0^{\lambda t}d\sigma\left[\left(\frac{\partial P}{\partial\sigma}\right)_{\sigma=0}+\frac{1}{2}\left(\frac{\partial^2 P}{\partial\sigma^2}\right)_{\sigma=0}\left(\frac{\sigma}{\lambda}\right)-\frac{\partial}{\partial t}\right]N(t,\mathbf{x})\left(\frac{\sigma}{\lambda}\right) \times\exp(-\sigma)=0, \qquad (14)$$

where we neglected the terms $\sim O[(\sigma/\lambda)^3]$. Then Eq. (14) reduces to

$$\frac{\partial N(t,\mathbf{x})}{\partial t}=\left(\frac{\partial P}{\partial\sigma}+\frac{1}{\lambda}\frac{\partial^2 P}{\partial\sigma^2}\right)_{\sigma=0}N(t,\mathbf{x}), \qquad (15)$$

where the operator $P(\sigma,\mathbf{x},i\boldsymbol{\nabla})$ is given by

$$P(\sigma,\mathbf{x},i\boldsymbol{\nabla})=1-\sigma[\langle b\rangle+\langle v_m\rangle\nabla_m+D\Delta]+(\sigma^2/2)[\nabla_p\langle v_p b\rangle + \nabla_p\langle v_p v_m\rangle\nabla_m+\langle v_m b\rangle\nabla_m]+O(\sigma^3). \qquad (16)$$

Substituting Eq. (16) into Eq. (15), we obtain

$$\frac{\partial N}{\partial t}+\nabla_m(V_m^{(\mathrm{eff})}N-\hat{D}_{mn}\nabla_n N)=0, \qquad (17)$$

where

$$\mathbf{V}_m^{(\mathrm{eff})}=\mathbf{U}-\langle\tau\mathbf{v}b\rangle \qquad (18)$$

$$\hat{D}_{mn}=D\delta_{mn}+\langle\tau v_m v_n\rangle, \qquad (19)$$

and $\langle\mathbf{v}\rangle=\mathbf{U}$ and $\langle b\rangle=\boldsymbol{\nabla}\cdot\mathbf{U}$. The turbulent transport coefficients [see Eqs. (18) and (19)] derived above are different by a factor 2 from those obtained for a model with a δ correlated in time random velocity field, and for a model with a constant renewal time. It is known that the model with a δ correlated in time random velocity field and the model with a constant renewal time do not recover the known turbulent diffusion tensor $\langle\tau v_m v_n\rangle$. Thus, in these two models, one has to rescale $\tau\to 2\tau$ in order to recover a correct coefficient of turbulent diffusion. The reason for this inconsistency is purely formal. Thus, the δ function is normalized in the interval from $-\infty$ to ∞, while in the model with a δ correlated in time random velocity field one has to integrate over the time from 0 to ∞. The reason that the coefficient of turbulent diffusion in a turbulent velocity field with random renewals is twice as large in comparison with that for a constant renewal time is as follows. Let us consider two random walks. The first one, $\zeta_n^{(1)}$, performs jumps at a distance $v\tau$ to the left and right with probabilities 1/2 (here v and τ are constants). The second one, $\zeta_n^{(2)}$, performs jumps to the left and right with the same probabilities with velocity v during a random time interval $\tilde{\mu}$, where $\tilde{\mu}$ has a Poisson distribution with a mean value τ. A direct calculation shows that the root mean square value of $\zeta_n^{(2)}$ is twice as large as the root mean square value of $\zeta_n^{(1)}$. Formally, a factor 2 in $\langle\tau v_m v_n\rangle$ and in $\langle\tau\mathbf{v}b\rangle$ arises because these terms in operator P are proportional to σ^2 [see Eq. (16)] and the integral $\int_0^\infty\sigma^2\exp(-\sigma)\,d\sigma=2$.

Note that the turbulent diffusion D_{mn} and the effective drift velocity $\mathbf{V}_m^{(\mathrm{eff})}$ in the case $\tau\ll\min\{\tau_c,\tau_0\}$ are determined only by the renewal time τ, rather than by τ_c or τ_0. The reason is that for the small renewal time $\tau \ll \min\{\tau_c;\tau_0\}$ the velocity field does not change during a small time τ, and the turbulent transport coefficients are determined by the renewal time τ only.

B. Random velocity field with Gaussian statistics for the integrals $\int \mathbf{v}(\mu,\boldsymbol{\xi}) d\mu$ and $\int b(\mu,\boldsymbol{\xi})\, d\mu$

Now we consider a model with a random homogeneous velocity field in which the integrals $\int \mathbf{v}(\mu,\boldsymbol{\xi})\, d\mu$ and $\int b(\mu,\boldsymbol{\xi})\, d\mu$ have Gaussian statistics. Using an identity $E\{\exp(a\eta)\}=\exp(a^2/2)$ and Eq. (11), we obtain

$$P(\sigma,\mathbf{x},i\boldsymbol{\nabla})=\exp[(1/2)\sigma\hat{D}_{mn}\nabla_m\nabla_n-\sigma\mathbf{V}^{(\mathrm{eff})}\cdot\boldsymbol{\nabla}], \quad (20)$$

$$\sigma V_m^{(\mathrm{eff})}=-M_{\boldsymbol{\xi}}\{\langle \xi_m^*(\mathbf{x})g(\mathbf{x})\rangle\}, \quad (21)$$

$$\sigma\hat{D}_{mn}=M_{\boldsymbol{\xi}}\{\langle \xi_m^*(\mathbf{x})\xi_n^*(\mathbf{x})\rangle\}, \quad (22)$$

where η is a Gaussian random variable with zero mean and unit variance, $G=\exp(g)$, and $M_{\boldsymbol{\xi}}\{\langle G\rangle\}=1$. The latter yields $M_{\boldsymbol{\xi}}\{\langle g\rangle\}=-(1/2)M_{\boldsymbol{\xi}}\{\langle \hat{g}^2\rangle\}$, where $g=\langle g\rangle+\tilde{g}$. When $\tau_c \ll \tau$ or $\tau_c \ll \tau_N$, these turbulent transport coefficients are given by

$$V_m^{(\mathrm{eff})}=-2\int_0^\infty M_{\boldsymbol{\xi}}\{\langle v_m(0,\boldsymbol{\xi})b(\sigma',\boldsymbol{\xi})\rangle\}\, d\sigma'-M_{\boldsymbol{\xi}}\{\langle \boldsymbol{\xi}\rangle\}/\sigma, \quad (23)$$

$$D_{mn}=2\int_0^\infty M_{\boldsymbol{\xi}}\{\langle v_m(0,\boldsymbol{\xi})v_n(\sigma',\boldsymbol{\xi})\rangle\}\, d\sigma', \quad (24)$$

where $D_{mn}=D_{mn}+2D\delta_{mn}$, and we used the identity

$$M_{\boldsymbol{\xi}}\left\{\left\langle \int_0^\mu a_m(\mu',\mathbf{x})\, d\mu' \int_0^\mu c_n(\mu'',\mathbf{x})\, d\mu''\right\rangle\right\}$$
$$\simeq 2\mu\int_0^\infty M_{\boldsymbol{\xi}}\{\langle a_m(0,\mathbf{x})c_n(\mu',\mathbf{x})\rangle\}\, d\mu'. \quad (25)$$

Note that $M_{\boldsymbol{\xi}}\{\langle \boldsymbol{\xi}^*\rangle\}=-M_{\boldsymbol{\xi}}\{\langle \int_0^\tau \mathbf{v}(\sigma',\boldsymbol{\xi})\, d\sigma'\rangle \neq 0$, even when $\langle \mathbf{v}(t,\mathbf{x})\rangle=0$. When $\langle \mathbf{v}b\rangle \neq 0$ the mean trajectory $M_{\boldsymbol{\xi}}\{\langle \boldsymbol{\xi}^*\rangle\}\neq 0$, due to an anisotropy which is caused by the nonzero vector $\langle \mathbf{v}b\rangle$. Substituting Eqs. (20), (23), and (24) into Eq. (10), we obtain

$$\frac{\partial N}{\partial t}=[(1/2)\hat{D}_{mn}\nabla_m\nabla_n-\mathbf{V}^{(\mathrm{eff})}\cdot\boldsymbol{\nabla}]N(t,\mathbf{x}). \quad (26)$$

Using Eqs. (13) and (20) for the case $\max\{\tau_c;\tau_N\}\ll\tau$ also yields Eq. (26). Note that the form of Eq. (26) obtained for $\tau_c\ll\min\{\tau,\tau_N\}$ and $\max\{\tau_c;\tau_N\}\ll\tau$ coincides with Eq. (17), derived for the case $\tau\ll\min\{\tau_c;\tau_N\}$.

C. Random velocity field with Gaussian statistics for the Lagrangian trajectories $\boldsymbol{\xi}^*$ and random function $\boldsymbol{\xi}^* G(t,\boldsymbol{\xi})$

Now we consider a model for a random velocity field with Gaussian statistics for the Lagrangian trajectories, i.e., we assume that the Lagrangian trajectories $\boldsymbol{\xi}^*$ and the random function $\boldsymbol{\xi}^* G(t,\boldsymbol{\xi})$ have Gaussian statistics. For a homogeneous random velocity field, we assume that $M_{\boldsymbol{\xi}}\{\langle \boldsymbol{\xi}^*\rangle\}=0$, $M_{\boldsymbol{\xi}}\{\langle \xi_m^*\xi_n^*\rangle\}=W\delta_{mn}f(\sigma)$, $M_{\boldsymbol{\xi}}\{\langle \xi_m^* G\rangle\}=-\sigma V_m$, and $M_{\boldsymbol{\xi}}\{\langle G\rangle\}=1$. The latter equation implies the conservation of the total number of particles. Here $f(\sigma)=\sigma^2$ for $\sigma<\tau_c$ and $f(\sigma)=\tau_c^2(2\sigma/\tau_c-1)$ for $\sigma\geq\tau_c$. These equations yield

$$P(\sigma,\mathbf{x},i\boldsymbol{\nabla})=(1-\sigma\mathbf{V}\cdot\boldsymbol{\nabla})\exp[(1/2)f(\sigma)W\Delta], \quad (27)$$

where we used the identities

$$E\{\exp(a\eta)\}=\exp(a^2/2), \quad (28)$$

$$\left(\frac{\partial}{\partial\mu}\exp(a+\mu g)\right)_{\mu=0}=g\exp(a), \quad (29)$$

and η is a Gaussian random variable with zero mean and unit variance. Equations (10) and (29) yield, for $\tau_c\ll\tau$,

$$\frac{\partial N}{\partial t}=\hat{L}N(t,\mathbf{x}), \quad (30)$$

$$\hat{L}=\tau_c W\Delta+\frac{1}{2\tau}[\sqrt{1-4\tau\mathbf{V}\cdot\boldsymbol{\nabla}}-1]. \quad (31)$$

When $\tau_c\ll\tau\ll\tau_N$, Eq. (31) reads

$$\hat{L}=\tau_c W\Delta-\mathbf{V}\cdot\boldsymbol{\nabla}-\tau(\mathbf{V}\cdot\boldsymbol{\nabla})^2, \quad (32)$$

where $W/V^2>\tau/\tau_c$ and $\tau_N=\min\{L_N/V;L_N^2/D_T\}$. For $\tau\gg\tau_N$, the mean number density of particles is determined by the equation

$$N(t,\mathbf{x})=(1-t\mathbf{V}\cdot\boldsymbol{\nabla})\exp[t\tau_c W\Delta]N(t=0,\mathbf{x}). \quad (33)$$

In order to derive Eq. (33), we used Eq. (10) in which the first term on the right hand side is neglected for large τ (small λ). Now we differentiate Eq. (33) with respect to t, and neglect terms $\sim O(t^{-1})$ for very large times. Thus we arrive at the equation

$$\frac{\partial N}{\partial t}=\tau_c W\Delta N(t,\mathbf{x}). \quad (34)$$

Note that for this model of the random velocity field, the form of a mean-field equation for a passive scalar is different from that derived for other models [see Eqs. (30) and (31)]. Moreover, in the case of $\tau\gg\max\{\tau_c;\tau_N\}$, the mean-field equation for a passive scalar does not contain the effective velocity $\mathbf{V}^{(\mathrm{eff})}$, i.e., it is independent of the compressibility of the velocity field $\mathbf{v}$. Only in the case $\tau_c\ll\tau\ll\tau_N$ the mean-field equation for a passive scalar is similar to Eqs. (17) and (26), which were derived using different models for random velocity field. Indeed, Eqs. (30) and (32) for $\tau_c\ll\tau\ll\tau_N$ yield

$$\frac{\partial N}{\partial t}+\nabla_m(V_m N-\tilde{W}_{mn}\nabla_n N)=0, \quad (35)$$

where $\tilde{W}_{mn}=\tau_c W\delta_{mn}-\tau V_m V_n$.

D. Weak inhomogeneity and slow evolution of the mean number density of particles

In the case $\tau_c \ll \tau \ll \tau_N$ the mean-field equation for a passive scalar is independent of the statistics of the velocity field, Lagrangian trajectories and other characteristics (where $\tau_N = \min\{L_N/V; L_N^2/D_T\}$). Indeed, consider a weakly inhomogeneous case and slow evolution of the mean number density of particles N. Note that a weakly inhomogeneous case implies large τ_N. Expanding in Taylor series the first term of Eq. (12) and Eq. (11) for small spatial and time derivatives of N, we arrive at the equation

$$\frac{\partial N}{\partial t} + (\mathbf{V}^{(\mathrm{eff})}\boldsymbol{\nabla} - \tilde{D}_{mn}\nabla_m\nabla_n)N = 0, \tag{36}$$

where $\tilde{D}_{mn} = M_{\xi}\{\langle G\xi_m^*\xi_n^*\rangle\}/2\sigma$ and $\mathbf{V}^{(\mathrm{eff})} = -M_{\xi}\{\langle G\boldsymbol{\xi}^*\rangle\}/\sigma$. Note that the correlation functions $M_{\xi}\{\langle G\xi_m^*\xi_n^*\rangle\}$ and $M_{\xi}\{\langle G\boldsymbol{\xi}^*\rangle\}$ are proportional to σ. Here we neglected the small second term $\sim \exp(-\tau_N/\tau)$ in Eq. (12). Notably, in the above derivation we did not use any assumptions about the statistics of the velocity field, the Lagrangian trajectories, and the function G.

IV. DISCUSSION

We developed a mean-field theory of transport of a passive scalar (e.g., particles and gases) in a random velocity field with a finite correlation time τ_c. We used a model of the velocity field with a random renewal time. The mean-field equation for a passive scalar is derived for arbitrary ratios τ/τ_c, where τ is the mean renewal time. In a general case the latter equation is an integral equation. However, in various applications the second-order differential equations (in spatial variables) for the mean passive scalar field are used. We determined the conditions when the mean-field equation can be reduced to a second-order differential equation for different models of a turbulent velocity field (i.e., different statistics of the velocity field, Lagrangian trajectories, and other flow field characteristics). We demonstrated an important role of the statistics of the Lagrangian trajectories of particles for turbulent transport of a passive scalar. We also found that the turbulent diffusion coefficient is determined by the minimum of the correlation τ_c and average renewal τ times.

The considered model of a turbulent velocity field with a random renewal time is quite general. In particular, when a mean renewal time is very large in comparison with the correlation time of the velocity field, we recover the results obtained for a turbulent flow without renewal. On the other hand, in the case $\tau \ll \tau_c$ the random velocity field behaves similarly to the δ correlated in time random velocity field.

It was recently found (see Ref. [19]) that due to the finite correlation time of a random velocity field, the field of Lagrangian trajectories is compressible even if the velocity field is incompressible. The compressibility of the field of Lagrangian trajectories results in the excitation of a small-scale instability of the second moment of the particle number density, and the formation of small-scale inhomogeneities of fluctuations of the particle number density, even for a very small compressibility of a random velocity field with a finite correlation time (see Ref. [19]).

The obtained results are important in studies of turbulent transport in various naturally occurring and industrial flows, where commonly used local second-order differential transport models are employed. In particular, these results may be of relevance in atmospheric phenomena (e.g., dynamics of cloud droplets and atmospheric aerosols, smog formation, etc). We considered a low Mach numbers compressible turbulent fluid flow $\boldsymbol{\nabla}\cdot\mathbf{v} \neq 0$, i.e., $\boldsymbol{\nabla}\cdot\mathbf{v} = -(\mathbf{v}\cdot\boldsymbol{\nabla}\rho)/\rho$, where ρ is the density of the fluid. In an atmosphere without temperature inversion, the characteristic density stratification length $\Lambda_\rho = |\boldsymbol{\nabla}\rho/\rho|^{-1} \simeq 8$ kilometers, and therefore $\boldsymbol{\nabla}\cdot\mathbf{v}$ is small. However, in an atmosphere with temperature inversion, the characteristic density stratification length in the vicinity of the temperature inversion layer $\Lambda_\rho \sim \Lambda_T \simeq 250$–$300$ m, where $\Lambda_T = |\boldsymbol{\nabla}T/T|^{-1}$. This implies that in the vicinity of the temperature inversion layer, the value of velocity divergence $\boldsymbol{\nabla}\cdot\mathbf{v}$ is not so small. The velocity field $\mathbf{v}_p$ of droplets and aerosols is also compressible, i.e., $\boldsymbol{\nabla}\cdot\mathbf{v}_p = \boldsymbol{\nabla}\cdot\mathbf{v} + \tau_p\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}P_f/\rho) + O(\tau_p^2)$ Ref. [6], where the velocity of droplets or aerosols $\mathbf{v}_p$ depends on the velocity of the atmospheric fluid $\mathbf{v}$, and can be determined from the equation of motion for a particle. This equation represents a balance of particle inertia, with the fluid drag force produced by the motion of the particle relative to the atmospheric fluid and gravity force. Here P_f is the atmospheric fluid pressure, and τ_p is the characteristic time of the coupling between the particle and atmospheric fluid (Stokes time). Solution of the equation of motion for small particles with $\rho_p \gg \rho$ yields $\mathbf{v}_p = \mathbf{v} + \mathbf{V}_g - \tau_p[\partial\mathbf{v}/\partial t + ((\mathbf{v}+\mathbf{V}_g)\cdot\boldsymbol{\nabla})\mathbf{v}] + O(\tau_p^2)$ (see Ref. [30]), where $\mathbf{V}_g = \tau_p\mathbf{g}$ is the terminal fall velocity, $\mathbf{g}$ is the acceleration due to gravity, and ρ_p is the material density of particles.

The formation of large-scale inhomogeneities of aerosols and droplets in a turbulent atmosphere is associated with both particle inertia and correlation between velocity and temperature fluctuations of an atmospheric fluid in the presence of a nonzero mean fluid temperature gradient Ref. [6]. Indeed, the inertia of particles results in the fact that particles with $\rho_p \gg \rho$ inside the turbulent eddy are carried out to boundary regions between the eddies by inertial forces (i.e., regions with low vorticity or high strain rate; see, e.g., Refs. [30,31]). On the other hand, the inertia effect causes $\boldsymbol{\nabla}\cdot\mathbf{v}_p \propto \tau_p\Delta P_f \neq 0$. In addition, for large Peclet numbers $\boldsymbol{\nabla}\cdot\mathbf{v}_p \propto -dn/dt$ [see Eq. (4)]. Therefore, $dn/dt \propto -\tau_p\Delta P_f$. This implies that in regions where $\Delta P_f < 0$, there is an accumulation of inertial particles (i.e., $dn/dt > 0$). Similarly, there is an outflow of inertial particles from the regions with $\Delta P_f > 0$. In a turbulence without large-scale external gradients of temperature, a drift from regions with increased (decreased) concentrations of inertial particles by a turbulent flow of fluid is equiprobable in all directions. The location of these regions is not correlated with the turbulent velocity field. Therefore they do not contribute to the large-scale flow of inertial particles.

The situation is drastically changed when there is a large-scale inhomogeneity of the temperature of the turbulent flow. In this case the mean heat flux $\langle \mathbf{v}\theta \rangle \neq 0$. Therefore, fluctuations of both the temperature θ and velocity $\mathbf{v}$ of the fluid are correlated. Fluctuations in the temperature cause fluctuations of the pressure of the fluid, and vice versa (see the equation of state $P_f = \rho \kappa_B T_f / m_\mu$, where κ_B is the Boltzmann constant, m_μ is the mass of molecules of atmospheric fluid, and T_f is the fluid temperature). The pressure fluctuations result in fluctuations of the number density of inertial particles. Indeed, an increase (decrease) in the pressure of the atmospheric fluid is accompanied by an accumulation (outflow) of the particles. Therefore, the direction of the mean flux of particles coincides with that of the heat flux, i.e., $\langle \mathbf{v}_p n \rangle \propto \langle \mathbf{v}\theta \rangle \propto -\boldsymbol{\nabla} T$, where $T = \langle T_f \rangle$ is the mean temperature of an atmospheric fluid with a characteristic value T_*, and $T_f = T + \theta$. Therefore the mean flux of the inertial particles (aerosols and droplets) is directed to the minimum of the mean temperature, and the inertial particles are accumulated in this region (e.g., in the vicinity of the temperature inversion layer). This effect is more pronounced when the atmospheric turbulent fluid flow is inhomogeneous in the direction of the mean temperature gradient. Thus, in a turbulent atmosphere, this effect (the effect of turbulent thermal diffusion) causes an accumulation of aerosol particles in the vicinity of the temperature inversion. On the other hand, turbulent diffusion results in a relaxation of the particle inhomogeneities. Thus two competitive mechanisms of particle transport, i.e., mixing by turbulent diffusion and the accumulation of particles due to turbulent thermal diffusion, exist simultaneously with the effect of gravitational settling of particles.

The effective velocity $\mathbf{V}^{(\mathrm{eff})}$ (which determines accumulation of particles caused by the effect of turbulent thermal diffusion) can be estimated as

$$\mathbf{V}^{(\mathrm{eff})} = -\langle \tau \mathbf{v}_p (\boldsymbol{\nabla} \cdot \mathbf{v}_p) \rangle = -(2/3) V_g A(\mathrm{Re}, a_*) \Lambda_P \ln(\mathrm{Re}) (\boldsymbol{\nabla} T)/T \qquad (37)$$

(see Refs. [6,18]), where we neglected the small molecular flux of the particles and the compressibility of the surrounding fluid. Here $A(\mathrm{Re}, a_*) = 1$ when the particle size $a_* < a_{\mathrm{cr}}$, $A(\mathrm{Re}, a_*) = 1 - 3\ln(a_*/a_{\mathrm{cr}})/\ln(\mathrm{Re})$ for $a_* \geqslant a_{\mathrm{cr}}$, $a_{\mathrm{cr}} = r_d (\rho/\rho_p)^{1/2}$, $\mathrm{Re} = l_0 u_0 / \nu$ is the Reynolds number, $r_d = l_0 \mathrm{Re}^{-3/4}$ is the viscous scale of the turbulent fluid flow, and $D_T = u_0 l_0 / 3$ is the coefficient of the turbulent diffusion. Thus, e.g., $a_{\mathrm{cr}} \sim 20$ μm for $\mathrm{Re} = 10^7$, $l_0 = 100$ m, and $\rho_p = 1$ g/cm^3. The effective velocity $\mathbf{V}^{(\mathrm{eff})}$ of particles determines a turbulent contribution to the particle velocity, due to both the effect of inertia and the mean temperature gradient. Remarkably, Eq. (37) for the effective velocity of the particles provides a local parametrization of the turbulence effects, and it can be directly incorporated into existing atmospheric numerical models. It is seen from Eq. (37) that the ratio $|\mathbf{V}^{(\mathrm{eff})}/V_g|$ is of the order of

$$|\mathbf{V}^{(\mathrm{eff})}/V_g| \sim (\Lambda_P/\Lambda_T)(\delta T/T_*) \ln \mathrm{Re},$$

where δT is the temperature difference in the scale Λ_T, and T_* is the characteristic temperature. In an atmosphere without temperature inversion, the temperature gradient is 1 K per 1000 m, and the ratio $|\mathbf{V}^{(\mathrm{eff})}/V_g|$ for particles of radius 30–300 μm changes in the interval: 0.27–0.15. In an atmosphere without temperature inversion the effective particle velocity is directed opposite to the terminal fall velocity, and thus the effective particle velocity decreases the effective sedimentation velocity by 10–30 %. On the other hand, in an atmosphere with a temperature inversion the temperature gradient is 1 K per 100 m and the ratio $|\mathbf{V}^{(\mathrm{eff})}/V_g|$ for particles of radius $a_* = 30$–300 μm changes in the interval 2.7–1.5. In this case, the effective particle velocity $\mathbf{V}^{(\mathrm{eff})}$ is larger than the terminal fall velocity V_g. In the atmosphere with a temperature inversion, the effective particle velocity $\mathbf{V}^{(\mathrm{eff})}$ is directed to the temperature minimum, and this results in an accumulation of particles in the vicinity of the temperature inversion.

The additional turbulent nondiffusive flux of particles due to the effective velocity $\mathbf{V}^{(\mathrm{eff})}$ results in the formation of inhomogeneities of aerosol distributions, whereby the initial spatial distribution of particles in the turbulent atmosphere evolves under certain conditions into a large-scale inhomogeneous distribution due to the excitation of an instability. One of the most important conditions for the instability is an inhomogeneous spatial distribution of the mean atmospheric temperature (see Refs. [6,17,28]). In particular, the instability can be excited in the vicinity of the minimum in the mean temperature. The characteristic time of the formation of inhomogeneities of particles is $\tau_f \sim \Lambda_T / |V^{(\mathrm{eff})} - V_g|$. The formation of inhomogeneities is possible when $V^{(\mathrm{eff})} > V_g$. The initially spatial distribution of the concentration of the inertial particles evolves into a pattern containing regions with increased (decreased) concentrations of particles. The characteristic vertical size of the inhomogeneity is of the order of

$$l_f \sim \Lambda_T \left[\left(\frac{V_g \Lambda_P}{D_T} \right) \left(\frac{\delta T}{T_*} \right) \ln \mathrm{Re} \right]^{-1/2}.$$

Thus it is important to take into account the additional turbulent nondiffusive flux of particles due to the effective velocity $\mathbf{V}^{(\mathrm{eff})}$ in atmospheric phenomena (e.g., atmospheric aerosols, cloud formation, and smog formation). Observations of vertical distributions of aerosols in the atmosphere show that maximum concentrations can occur within temperature inversion layers (see, e.g., Ref. [32], and references therein). Using the characteristic parameters of the atmospheric turbulent boundary layer (a maximum scale of turbulent flow $l_0 \sim 10^3$–10^4 cm; a velocity on a scale l_0 of $u_0 \sim 30$–100 cm/s; and a Reynolds number $\mathrm{Re} \sim 10^6$–10^7) we obtain that for particles with material density $\rho_p \sim 1-2\,\mathrm{g/cm}^3$ and radius $a_* = 30$ μm, the characteristic time of formation of inhomogeneities of the order of 11 min for the temperature gradient 1 K/100 m and 106 min for the temperature gradient 1 K/200 m. For particles of the size $a_* = 100$ μm, the characteristic time for the formation of inhomogeneities of the order of 1 min for the temperature gradient 1 K/100 m and 121 min for the temperature gradient 1 K/200 m. These estimates are in compliance with the characteristic times of formation of inhomogeneous structures in atmosphere. We expect that the spatial density $\mathrm{m}_p n$

of particles inside the inhomogeneous structures is of the order of the density ρ of the surrounding fluid.

The effect of turbulent thermal diffusion may also be of relevance in combustion. In particular, this effect may cause the formation of inhomogeneities in the spatial distribution of fuel droplets in internal combustion engines (see, e.g., Refs. [22,24,33]). Indeed, the characteristic parameters of turbulence in a cylinder of internal combustion engine are as follows: maximum scale of turbulent flow $l_0 \sim 0.5$–1 cm; a velocity on the l_0 scale of $u_0 \sim 100$ cm/s; a Reynolds number $\mathrm{Re} \sim (0.7-7)\times 10^3$; characteristic values of the mean temperature distribution on a scale $\Lambda_T \sim 13$–18 cm; and a dimensionless mean spatial temperature variation $\delta T/T_* \sim 0.3$–0.5 (see, e.g., Refs. [22,24,33]). Then the characteristic time of formation of inhomogeneities in a spatial distribution of droplets of radius $a_* = 30$ μm is $\sim(3$–$6)\times 10^{-2}$ s. Notably, this time is comparable to the duration of an engine cycle. These turbulence induced inhomogeneities in the spatial distribution of the evaporating fuel droplets have strong effects upon combustion, soot, and emission formation (see, e.g., Refs. [22,24,33]).

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APPENDIX A: DERIVATION OF EQ. (5)

In this appendix we will derive Eq. (5). To this end we use an exact solution of Eq. (4) with an initial condition $n(t=s,\mathbf{x}) = n(s,\mathbf{x})$ in the form of the Feynman-Kac formula:

$$n(t,\mathbf{x}) = M_{\xi}\{G(t,s,\boldsymbol{\xi}(t,s))\, n(s,\boldsymbol{\xi}(t,s))\}, \tag{A1}$$

$$G(t,s,\boldsymbol{\xi}) = \exp\left[-\int_s^t b(\mu,\boldsymbol{\xi}(t,\mu))\, d\mu\right], \tag{A2}$$

where $M_{\xi}\{\cdot\}$ denotes the mathematical expectation over the Wiener paths $\boldsymbol{\xi}(t,s) = \mathbf{x} - \int_0^{t-s} \mathbf{v}[t-\mu,\boldsymbol{\xi}(t,\mu)]\, d\mu + (2D)^{1/2}\mathbf{w}(t-s)$. Now we assume that

$$n(t,\boldsymbol{\xi}) = \int \exp(i\boldsymbol{\xi}\cdot\mathbf{q})\, n(s,\mathbf{q})\, d\mathbf{q}. \tag{A3}$$

Substituting Eq. (A3) into Eq. (A2), we obtain

$$n(t,\mathbf{x}) = \int M_{\xi}\{G(t,s,\boldsymbol{\xi}(t,s))\exp[i\boldsymbol{\xi}^*\cdot\mathbf{q}]\, n(s,\mathbf{q})\} \times \exp(i\mathbf{q}\cdot\mathbf{x})\, d\mathbf{q}, \tag{A4}$$

where $\boldsymbol{\xi}^* = \boldsymbol{\xi} - \mathbf{x}$. In Eq. (A4), we expand the function $\exp[i\boldsymbol{\xi}^*\cdot\mathbf{q}]$ in Taylor series at $\mathbf{q}=0$, i.e.,

$$\exp[i\boldsymbol{\xi}^*\cdot\mathbf{q}] = \sum_{k=0}^{\infty} (1/k!)(i\boldsymbol{\xi}^*\cdot\mathbf{q})^k. \tag{A5}$$

We use an identity $(i\mathbf{q})^k \exp[i\mathbf{x}\cdot\mathbf{q}] = \boldsymbol{\nabla}^k \exp[i\mathbf{x}\cdot\mathbf{q}]$. Using this identity, and Eqs. (A4) and (A5) we obtain

$$n(t,\mathbf{x}) = M_{\xi}\left\{ G(t,s,\boldsymbol{\xi})\left[\sum_{k=0}^{\infty} (1/k!) \times (\boldsymbol{\xi}^*\cdot\boldsymbol{\nabla})^k\right] \int n(s,\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{x})\, d\mathbf{q}\right\}. \tag{A6}$$

After the inverse Fourier transformation in Eq. (A6), we obtain Eq. (5). Equation (A3) can be formally considered as an inverse Fourier transformation of the function $n(t,\boldsymbol{\xi})$. However, $\boldsymbol{\xi}$ is the Wiener path, which is not a usual spatial variable. Therefore, it is desirable to derive Eq. (A4) by a more rigorous method, as done in Appendix B.

APPENDIX B: DERIVATION OF EQ. (A4)

In this section we will derive Eq. (A4). In order to derive Eq. (A4) more rigorously we use an exact solution of the Cauchy problem for Eq. (4) with an initial condition $n(t=s,\mathbf{x}) = n(s,\mathbf{x})$ in another form,

$$n(t,\mathbf{x}) = \zeta_{\zeta}\{J(t,s,\boldsymbol{\zeta})\tilde{G}(t,s,\boldsymbol{\zeta})\, n(s,\boldsymbol{\zeta}(t,s))\}, \tag{B1}$$

where

$$\tilde{G}(t,s,\boldsymbol{\zeta}) = \exp\left[-\int_s^t b(\mu,\boldsymbol{\zeta}(t,\mu))\, d\mu\right], \tag{B2}$$

$$J(t,s,\boldsymbol{\zeta}) = \exp\left[-(2D)^{-1/2}\int_0^{t-s} \mathbf{v}(t-\eta,\boldsymbol{\zeta}(t,\eta))\cdot d\mathbf{w}(\eta) - (4D)^{-1}\int_0^{t-s} \mathbf{v}^2(t-\eta,\boldsymbol{\zeta}(t,\eta))\, d\eta\right], \tag{B3}$$

$$\boldsymbol{\zeta}(t,s) = \mathbf{x} + (2D)^{1/2}(\mathbf{w}(t) - \mathbf{w}(s)), \tag{B4}$$

where $\mathbf{w}(t)$ is a Wiener process, $M_{\zeta}\{\cdot\}$ denotes the mathematical expectation over the Wiener paths $\boldsymbol{\zeta}$, and $b = \boldsymbol{\nabla}\cdot\mathbf{v}$. Note that there is a singularity in Eq. (B3) at $D\to 0$. However, this singularity in the final result is eliminated by a change of variables (see below) Refs. [10,34]. As follows from Cameron-Martin-Girsanov theorem, the transformation from Eq. (A1) to Eq. (B1) can be considered as a change of variables $\boldsymbol{\xi}\to\boldsymbol{\zeta}$ in the path integral [Eq. (A1)] (see, e.g., Ref. [35]). Note that the path-integral representation for the effective diffusion function of a passive scalar field for $b=0$ was suggested by Drummond in Ref. [36].

The difference between solutions (B1) and (A1) is as follows. The function $n(s,\boldsymbol{\xi}(t,s))$ in Eq. (A1) depends explicitly on the random velocity field $\mathbf{v}$ via the Wiener path $\boldsymbol{\xi}$, while the function $n(s,\boldsymbol{\zeta}(t,s))$ in Eq. (B1) is independent of the velocity $\mathbf{v}$. Using Eq. (B1) allows us to separate the average over the random Wiener process, and the average over the random velocity field. Due to the Markovian property of the Wiener process, solution (B1) can be rewritten in the form

$$n(t,\mathbf{x})=E\{S(t,s,\mathbf{x},\mathbf{Y})\,n(s,\mathbf{Y})\}=\int \tilde{Q}(t,s,\mathbf{x},\mathbf{y})n(s,\mathbf{y})\,d\mathbf{y}, \tag{B5}$$

where

$$\tilde{Q}(t,s,\mathbf{x},\mathbf{y})=(4\pi D(t-s))^{-3/2}\times\exp\left(-\frac{(\mathbf{y}-\mathbf{x})^2}{4D(t-s)}\right)S(t,s,\mathbf{x},\mathbf{y}), \tag{B6}$$

$$S(t,s,\mathbf{x},\mathbf{y})=M_{\boldsymbol{\nu}}\{J(t,s,\boldsymbol{\nu})\tilde{G}(t,s,\boldsymbol{\nu})\}, \tag{B7}$$

and $M_{\boldsymbol{\nu}}\{\cdot\}$ means a path integral taken over the set of Wiener trajectories $\boldsymbol{\nu}$ which connect points $(t,\mathbf{x})$ and $(s,\mathbf{y})$. The mathematical expectation $E\{\cdot\}$ in Eq. (B5) denotes an averaging over a set of random points $\mathbf{Y}$ which have Gaussian statistics. Here we used here the following property of the averaging over the Wiener process: $E\{M_\nu\{\cdot\}\}=M_\zeta\{\cdot\}$.

Now we make a change of variables $(\mathbf{x},\mathbf{y})\rightarrow(\mathbf{x},\mathbf{y}=\mathbf{z}+\mathbf{x})$ in Eq. (B5), i.e.,

$$\tilde{Q}(t,s,\mathbf{x},\mathbf{y})=\tilde{Q}(t,s,\mathbf{x},\mathbf{z}+\mathbf{x})=Q(t,s,\mathbf{x},\mathbf{z}). \tag{B8}$$

The Fourier transformation in Eq. (B5) yields

$$n(t,\mathbf{x})=\int\int Q(t,s,\mathbf{x},\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{z})\,d\mathbf{k}\times\int n(s,\mathbf{q})\exp[i\mathbf{q}\cdot(\mathbf{z}+\mathbf{x})]\,d\mathbf{q}\,d\mathbf{z}.$$

Since $\delta(\mathbf{k}+\mathbf{q})=(2\pi)^{-3}\int\exp[i(\mathbf{k}+\mathbf{q}\cdot\mathbf{z})]d\mathbf{z}$, we obtain

$$n(t,\mathbf{x})=(2\pi)^3\int Q(t,s,\mathbf{x},-\mathbf{q})n(s,\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{x})d\mathbf{q}. \tag{B9}$$

In Eq. (B9), the function $Q(t,s,\mathbf{x},-\mathbf{q})$ is given by

$$Q(t,s,\mathbf{x},-\mathbf{q})=(2\pi)^{-3}\int Q(t,s,\mathbf{x},\mathbf{z})\exp(i\mathbf{q}\cdot\mathbf{z})\,d\mathbf{z}. \tag{B10}$$

Substituting $\tilde{Q}(t,s,\mathbf{x},\mathbf{y})=Q(t,s,\mathbf{x},\mathbf{z})$ in Eq. (B5), and taking into account that $\mathbf{y}=\mathbf{z}+\mathbf{x}$ we obtain

$$n(t,\mathbf{x})=\int Q(t,s,\mathbf{x},\mathbf{z})n(s,\mathbf{z}+\mathbf{x})d\mathbf{z}. \tag{B11}$$

Equation (B10) can be rewritten in the form

$$(2\pi)^3Q(t,s,\mathbf{x},-\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{x})=\int Q(t,s,\mathbf{x},\mathbf{z})\exp[i\mathbf{q}\cdot(\mathbf{z}+\mathbf{x})]\,d\mathbf{z}. \tag{B12}$$

The right hand sides of Eqs. (B11) and (B12) coincide when $n(s,\mathbf{z}+\mathbf{x})=\exp[i\mathbf{q}\cdot(\mathbf{z}+\mathbf{x})]$. Thus a particular solution [Eq. (B11)] of Eq. (4) with the initial condition $n(s,\mathbf{y})=\exp(i\mathbf{q}\cdot\mathbf{y})$ coincides in form with integral (B12). On the other hand, a solution of Eq. (4) is given by Eq. (B1). Substituting the initial condition $n(s,\boldsymbol{\zeta})=\exp(i\mathbf{q}\cdot\boldsymbol{\zeta})=\exp[i\mathbf{q}\cdot(\mathbf{x}+(2D)^{1/2}\mathbf{w})]$ into Eq. (B1) we obtain

$$n(t,\mathbf{x})=M_\zeta\{J(t,s,\boldsymbol{\zeta})\tilde{G}(t,s,\boldsymbol{\zeta})\exp[i\mathbf{q}\cdot(\mathbf{x}+(2D)^{1/2}\mathbf{w})]\}. \tag{B13}$$

Comparing Eqs. (B11)–(B13), we find that

$$Q(t,s,\mathbf{x},-\mathbf{q})=(2\pi)^{-3}M_\zeta\{J(t,s,\boldsymbol{\zeta})\tilde{G}(t,s,\boldsymbol{\zeta})\times\exp[i(2D)^{1/2}\mathbf{q}\cdot\mathbf{w}]\}. \tag{B14}$$

Now we rewrite Eq. (B14) using the Feynman-Kac formula (A1). The result is given by

$$Q(t,s,\mathbf{x},-\mathbf{q})=(2\pi)^{-3}M_\xi\{G(t,s,\boldsymbol{\xi}(t,s))\exp[i\boldsymbol{\xi}^*\cdot\mathbf{q}]\}, \tag{B15}$$

where $\boldsymbol{\xi}^*=\boldsymbol{\xi}-\mathbf{x}$. Substituting Eq. (B15) into Eq. (B9) we obtain Eq. (A4). The above derivation proves the assumption (A3) is correct for a Wiener path $\boldsymbol{\xi}$.

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